

# A NOTE ON THE $\text{Sopfr}(n)$ FUNCTION.

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ABSTRACT. The  $\text{Sopfr}(n)$  function is defined as the sum of prime factors of  $n$  each of which is taken with its multiplicity. This function is studied numerically. The analogy between  $\text{Sopfr}(n)$  and the primes distribution function is drawn and some conjectures for prime numbers formulated in terms of the  $\text{Sopfr}(n)$  function are suggested.

## 1. INTRODUCTION.

The  $\text{Sopfr}(n)$  function is defined as the sum of prime factors of its positive integer argument  $n$  (see [1]). For  $n = 1$  this function is defined to be equal to zero:  $\text{Sopfr}(1) = 0$ . If  $n$  is prime, then  $\text{Sopfr}(n) = n$ . If  $n$  is a product of prime numbers

$$n = p_1^{k_1} \cdot \dots \cdot p_s^{k_s}, \quad (1.1)$$

then  $\text{Sopfr}(n)$  is calculated as the sum

$$\text{Sopfr}(n) = k_1 p_1 + \dots + k_s p_s. \quad (1.2)$$

Note that the prime factors  $p_1, \dots, p_s$  in the sum (1.2) are taken with their multiplicities  $k_1, \dots, k_s$  in the expansion (1.1). Therefore the function  $\text{Sopfr}(n)$  is similar to the logarithm. One can easily prove the following identity for it:

$$\text{Sopfr}(n_1 \cdot n_2) = \text{Sopfr}(n_1) + \text{Sopfr}(n_2). \quad (1.3)$$

The  $\text{Sopfr}(n)$  function is used in defining Ruth-Aaron pairs named after two famous baseball players [George Herman Ruth Jr.](#) and [Henry Louis Aaron](#) (see [2]). In mathematics a Ruth-Aaron pair is a pair of consecutive numbers  $n$  and  $n + 1$  whose sums of prime factors are equal to each other:

$$\text{Sopfr}(n) = \text{Sopfr}(n + 1). \quad (1.4)$$

The numbers 714 and 715 constitute the most famous Ruth-Aaron pair.

Let  $x$  be an integer number and let  $p$ ,  $q$ ,  $r$ , and  $s$  be four numbers expressed through  $x$  by the following four polynomials:

$$\begin{aligned} p &= 8x + 5, & q &= 48x^2 + 24x - 1, \\ r &= 2x + 1, & s &= 48x^2 + 30x - 1. \end{aligned} \quad (1.5)$$

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Using the formulas (1.5), one easily derives that

$$pq + 1 = 2^2 rs, \quad p + q = 2 \cdot 2 + r + s. \quad (1.6)$$

Due to (1.6) and (1.3), if  $p, q, r, s$  all are prime numbers, then the numbers  $n = pq$  and  $n + 1 = 4rs$  constitute a Ruth-Aaron pair, i. e. they satisfy the equality (1.4). Schinzel's H-conjecture (see [3], [4], and [5]) implies that there are infinitely many integer numbers  $x$  such that the numbers  $p, q, r$ , and  $s$  given by the polynomials (1.5) all are prime.

In this paper we treat  $\text{Sopfr}(n)$  as an analog of the primes distribution function  $\pi(n)$ . The value  $\pi(n)$  of this function is defined as the number of positive primes less than or equal to  $n$ . Gauss and Legendre (see [6]) in 1792–1808 conjectured the following asymptotic behavior of the function  $\pi(n)$ :

$$\pi(n) \sim \frac{n}{\ln(n)} \quad \text{as } n \rightarrow \infty. \quad (1.7)$$

In 1849 and in 1852 P. L. Chebyshev proved two propositions very close to (1.7). The proposition (1.7) itself was proved in 1896 by Hadamard [7] and Valée Poussin [8]. See [9] for the modern explanation of their proof.

The main goal of this paper is to study the function  $\text{Sopfr}(n)$  numerically and formulate some conjectures similar to (1.7) for this function.

## 2. THE AVERAGED $\text{Sopfr}(n)$ FUNCTION.

The  $\text{Sopfr}(n)$  function is quite irregular. Looking at its graph (see [1]), one can find that the values of  $\text{Sopfr}(n)$  resemble random numbers. In order to make them more regular we average them over intervals between two consecutive squares:

$$A(n) = \sum_{i=n^2+1}^{(n+1)^2} \frac{\text{Sopfr}(i)}{(n+1)^2 - n^2}. \quad (2.1)$$

The function  $A(n)$  in (2.1) is the averaged  $\text{Sopfr}(n)$  function. We study its values in two intervals  $1 \leq n \leq 998$  and  $1000 \leq n \leq 3161$ . The graph of the function (2.1) in the first interval  $1 \leq n \leq 998$  is shown in Fig. 2.1. It is presented by a sequence of points whose coordinates are rendered in logarithmic scale, i. e.  $A_n = (x_n, y_n)$ , where  $x_n = \ln(A(n))$  and  $y_n = \ln(n)$ .

Looking at Fig. 2.1 below, one can see that the points  $A_n$  with  $n \geq 122 \approx e^{4.8}$  are approximated by a straight line. We write the equation of this straight line as

$$x = \alpha y + \beta. \quad (2.2)$$

In order to calculate the parameters  $\alpha$  and  $\beta$  in (2.2) we use the root mean squares method. For this purpose we use the following quadratic deviation function:

$$F(\alpha, \beta) = \sum_{n=122}^{998} (x_n - \alpha y_n - \beta)^2. \quad (2.3)$$

The quadratic function (2.3) has exactly one minimum point  $\alpha = \alpha_{\min}$ ,  $\beta = \beta_{\min}$ .

This minimum point is determined by the following linear equations:

$$\frac{\partial F(\alpha, \beta)}{\partial \alpha} = 0, \quad \frac{\partial F(\alpha, \beta)}{\partial \beta} = 0. \quad (2.4)$$

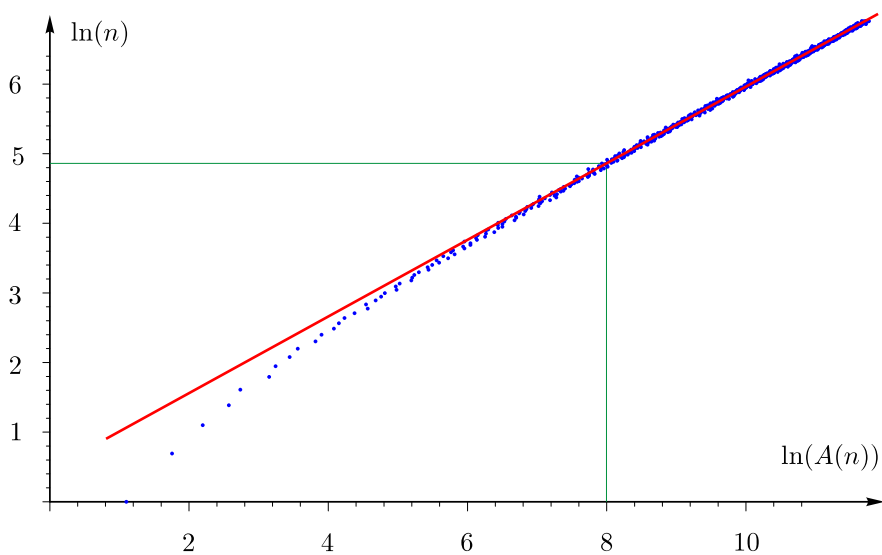


Fig 2.1

The function (2.3) and the equations (2.4) are computed numerically. Solving them,

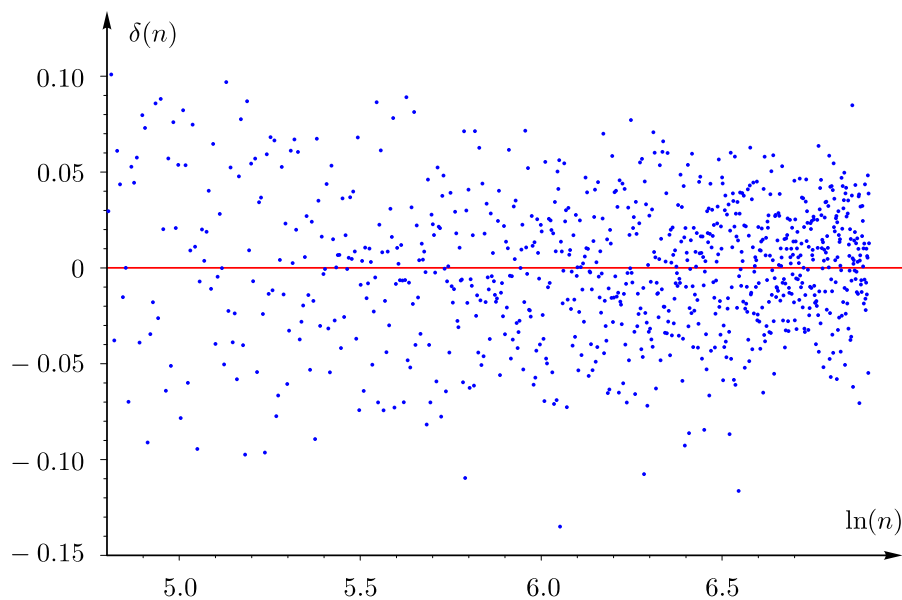


Fig 2.2

we find the numeric values of  $\alpha$  and  $\beta$  at the minimum of the function  $F(\alpha, \beta)$ :

$$\alpha \approx 1.820, \quad \beta \approx -0.847. \quad (2.5)$$

Having calculated the constants (2.5), now we draw the graph of the deviation function  $\delta(n) = \ln(A(n)) - \alpha \ln(n) - \beta$  in logarithmic scale. The graph of the function  $\delta(n)$  in Fig. 2.2 is presented by a series of points  $A_n = (x_n, y_n)$ , where  $x_n = \ln(n)$  and  $y_n = \delta(n)$ . Looking at this graph, we derive the following inequality for the deviation function  $\delta(n)$ :

$$-\delta_1 < \delta(n) < \delta_1, \text{ where } \delta_1 = 0.15 \text{ and } 122 \leq n \leq 998. \quad (2.6)$$

The next interval is  $1000 \leq n \leq 3161$ . The graph of the function (2.1) in this interval is shown in Fig. 2.3. Again it is presented by a sequence of points whose coordinates are rendered in logarithmic scale, i.e.  $A_n = (x_n, y_n)$ , where  $x_n = \ln(A(n))$  and  $y_n = \ln(n)$ . The graph in Fig. 2.3 is also approximated by a

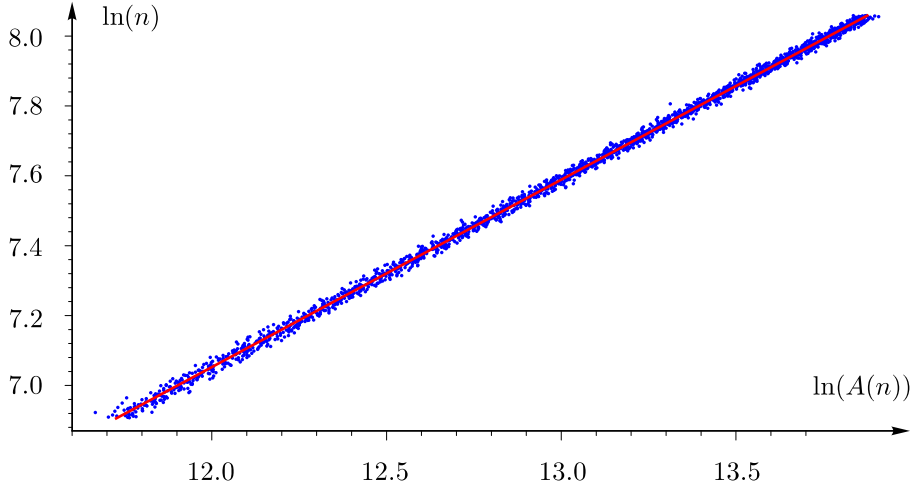


Fig 2.3

straight line. This straight line is given by the equation (2.2). The coefficients  $\alpha$  and  $\beta$  in this case are calculated by solving the equations (2.4) for the following quadratic deviation function, which is similar to (2.3):

$$F(\alpha, \beta) = \sum_{n=1000}^{3161} (x_n - \alpha y_n - \beta)^2. \quad (2.7)$$

The minimum of the function (2.7) corresponds to the following values of  $\alpha$  and  $\beta$ :

$$\alpha \approx 1.860, \quad \beta \approx -1.115. \quad (2.8)$$

The sharpness of the approximation of  $A(n)$  by the straight line in Fig. 2.3 is expressed through the deviation function  $\delta(n) = \ln(A(n)) - \alpha \ln(n) - \beta$ , where  $\alpha$

and  $\beta$  are given by the formulas (2.8):

$$-\delta_2 < \delta(n) < \delta_1, \text{ where } \delta_2 = 0.1 \text{ and } 1000 \leq n \leq 3161. \quad (2.9)$$

The inequalities (2.9) are similar to the above inequalities (2.6). They are derived by drawing the graph of the function  $\delta(n)$ . This graph is shown in Fig. 2.4 below.

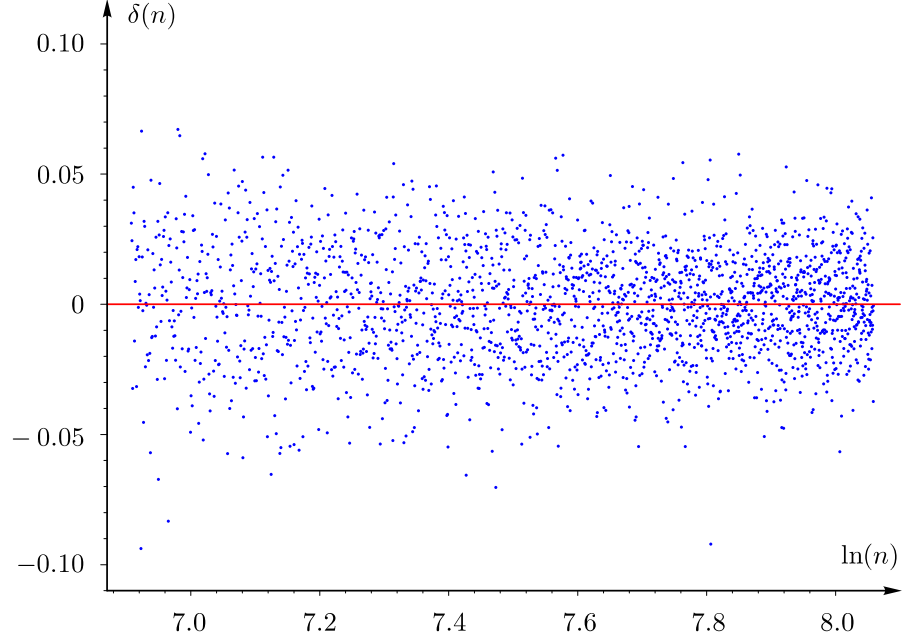


Fig 2.4

### 3. APPROXIMATION CONJECTURES.

Note that the parameter  $\alpha$  in (2.8) is greater than  $\alpha$  in (2.5). This means that the slope of the straight line approximating the graph of the function  $A(n)$  slightly grows as  $n \rightarrow \infty$ . To take into account this growth we replace the linear approximation in (2.2) by a nonlinear one. We choose the following formula for it

$$x = \alpha y + \beta + \gamma \ln(y) + \lambda e^{-y} + \mu e^{-2y}. \quad (3.1)$$

The choice of (3.1) means that  $A(n)$  is approximated by the formula

$$A(n) \approx B n^\alpha (\ln n)^\gamma \exp\left(\frac{\lambda}{n} + \frac{\mu}{n^2}\right), \text{ where } B = e^\beta. \quad (3.2)$$

In order to find the optimal values of the parameters  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\lambda$ , and  $\mu$  for the approximation (3.2) we apply the root mean squares method. Instead of (2.3) and (2.7) in this case we use the following deviation function:

$$F = \sum_{n=4}^{3161} (x_n - \alpha y_n - \beta - \gamma \ln y_n - \lambda e^{-y_n} - \mu e^{-2y_n})^2. \quad (3.3)$$

Remember that  $x_n = \ln(A(n))$  and  $y_n = \ln(n)$  in (3.3). The optimal values of  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\lambda$ , and  $\mu$  are determined by solving the equations

$$\frac{\partial F}{\partial \alpha} = 0, \quad \frac{\partial F}{\partial \beta} = 0, \quad \frac{\partial F}{\partial \gamma} = 0, \quad \frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial F}{\partial \mu} = 0. \quad (3.4)$$

The equations (3.4) are similar to the equations (2.4). Here is their solution:

$$\alpha \approx 2.001, \quad \beta \approx -0.047, \quad \gamma \approx -1.056, \quad \lambda \approx 1.187, \quad \mu \approx -2.240. \quad (3.5)$$

The exponential factor with  $\lambda$  and  $\mu$  in (3.2) is a decreasing function of  $n$ . For this reason we consider the function

$$B(n) = \frac{A(n)}{n^\alpha (\ln n)^\gamma}$$

and draw its graph. Like the graph of  $A(n)$ , it is presented as a sequence of points:

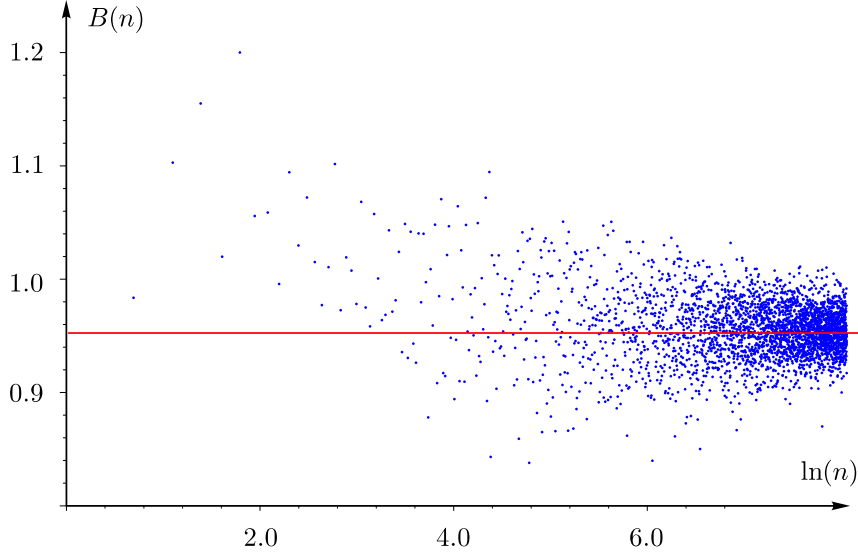


Fig 3.1

Looking at the graph in Fig. 3.1, we can formulate the following conjecture.

**Conjecture 3.1 (weak Sopfr( $n$ ) conjecture).** *There are four constants  $\alpha$ ,  $\gamma$ ,  $B_1$  and  $B_2$  such that the averaged Sopfr-function  $A(n)$  in (2.1) obey the inequalities*

$$B_1 n^\alpha (\ln n)^\gamma \leq A(n) \leq B_2 n^\alpha (\ln n)^\gamma \quad \text{for all } n > 1. \quad (3.6)$$

The graph points in Fig. 3.1 condense to a band as  $n \rightarrow \infty$ . Its width is restricted by the constants  $B_1$  and  $B_2$  in (3.6). The width of this band can vanish at infinity. For this option we can formulate the following conjecture.

**Conjecture 3.2 (strong Sopfr( $n$ ) conjecture).** *There are three constants  $\alpha$ ,  $\gamma$ , and  $B$  such that  $B > 0$  and the following condition is fulfilled:*

$$A(n) \sim B n^{\alpha} (\ln n)^{\gamma} \quad \text{as } n \rightarrow \infty. \quad (3.7)$$

Note that the constants  $\alpha$  and  $\gamma$  in (3.5) are very close to integer numbers. Therefore we can formulate another conjecture.

**Conjecture 3.3.** *The constants  $\alpha$  and  $\gamma$  either in (3.6) or in (3.7) are explicit numbers  $\alpha = 2$  and  $\gamma = -1$ .*

The averaged Sopfr( $n$ ) function (2.1) is similar to the primes distribution function. The above formulas (3.6) and (3.7) are similar to the formula (1.7).

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